Finite Wordlength Effects

- Finite register lengths and A/D converters cause errors in:-

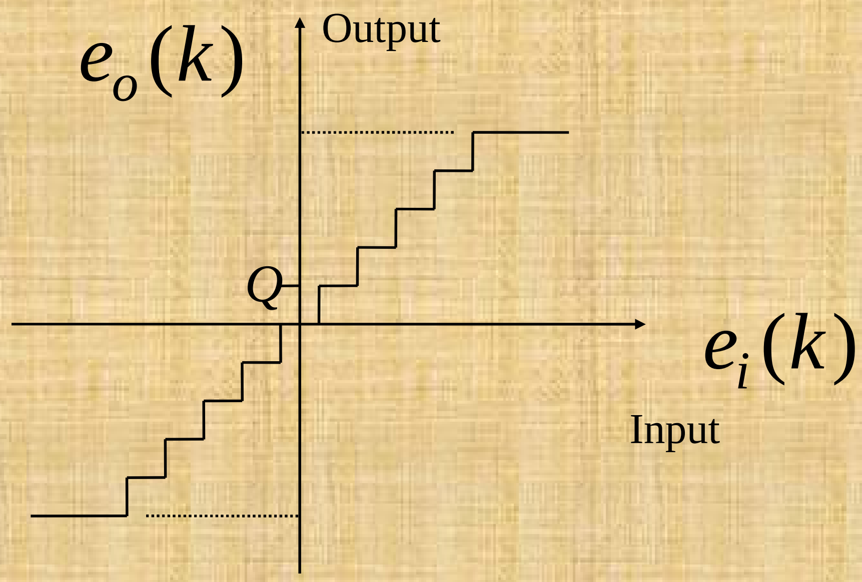
  (i) Input quantisation.

  (ii) Coefficient (or multiplier) quantisation

  (iii) Products of multiplication truncated or rounded due to machine length

Finite Wordlength Effects

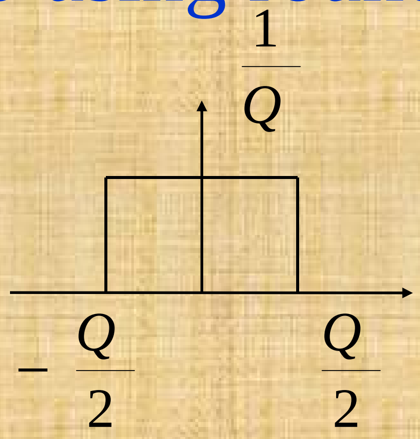
- Quantisation



$$-\frac{Q}{2} \le e_{i,o}(k) \le \frac{Q}{2}$$

Finite Wordlength Effects

- The pdf for *e* using rounding



- Noise power $$\sigma^2 = \int_{-Q/2}^{Q/2} e^2 p(e).de = E\{e^2\}$$

  or

$$\sigma^2 = \frac{Q^2}{12}$$

Finite Wordlength Effects

- Let input signal be sinusoidal of unity amplitude. Then total signal power $P = \frac{1}{2}$
- If b bits used for binary then $Q = 2/2^b$

  so that $\sigma^2 = 2^{-2b}/3$
- Hence $P/\sigma^2 = \frac{3}{2}.2^{+2b}$

  or $\text{SNR} = 1.8 + 6b$ dB

Finite Wordlength Effects

- Consider a simple example of finite precision on the coefficients a,b of second order system with poles $\rho e^{\pm j\theta}$

$$H(z)=\frac{1}{1-az^{-1}+bz^{-2}}$$

$$H(z)=\frac{1}{1-2\rho\cos\theta.z^{-1}+\rho^{2}.z^{-2}}$$

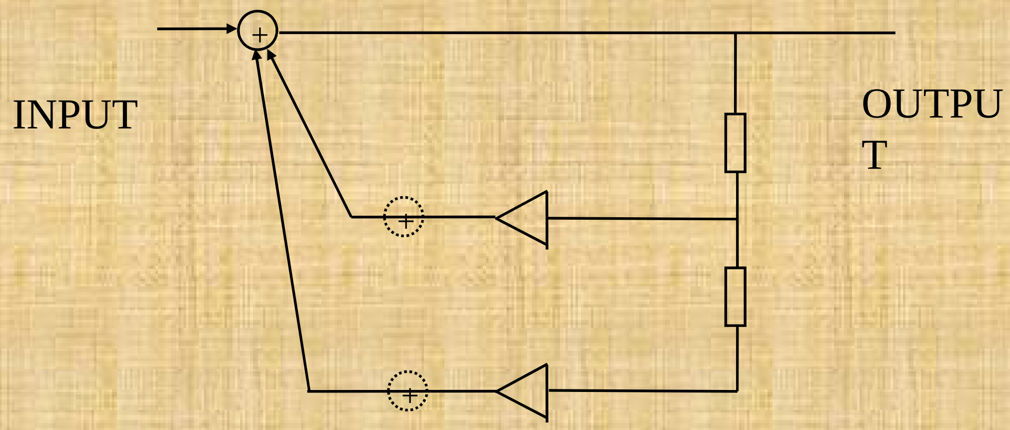
- where $a=2\rho\cos\theta$ $b=\rho^{2}$

Finite Wordlength Effects

bit pattern	$2\rho\cos\theta,\ \rho^2$	ρ
000	0	0
001	0.125	0.354
010	0.25	0.5
011	0.375	0.611
100	0.5	0.707
101	0.625	0.791
110	0.75	0.866
111	0.875	0.935
1.0	1.0	1.0

Finite Wordlength Effects

- Finite wordlength computations



dimple

Limit-cycles; "Effective Pole" Model; Deadband

- Observe that for $H(z) = 1 \Big/ (1 + b_1 z^{-1} + b_2 z^{-2})$
- instability occurs when $|b_2| \to 1$
- i.e. poles are
  - (i) either on unit circle when complex
  - (ii) or one real pole is outside unit circle.
- Instability under the "effective pole" model is considered as follows

Finite Wordlength Effects

- In the time domain with $H(z) = {}^{Y(z)}\!/_{X(z)}$
- $$y(n) = x(n) - b_1 y(n-1) - b_2 y(n-2)$$
- With $|b_2| \to 1$ for instability we have $Q[b_2 y(n-2)]$ indistinguishable from $y(n-2)$
- Where $Q[\cdot]$ is quantisation

Finite Wordlength Effects

- With rounding, therefore we have

$$b_2 y(n-2) \pm 0.5 \qquad\qquad y(n-2)$$

are indistinguishable (for integers)

or $$b_2 y(n-2) \pm 0.5 = y(n-2)$$

- Hence $$y(n-2) = \frac{\pm 0.5}{1-b_2}$$

- With both positive and negative numbers

$$y(n-2) = \frac{\pm 0.5}{1-|b_2|}$$

Finite Wordlength Effects

- The range of integers $\dfrac{\pm 0.5}{1-|b_2|}$

  constitutes a set of integers that cannot be individually distinguished as separate or from the asymptotic system behaviour.
- The band of integers $\left(-\dfrac{0.5}{1-|b_2|},\quad +\dfrac{0.5}{1-|b_2|}\right)$

  is known as the "deadband".
- In the second order system, under rounding, the output assumes a cyclic set of values of the deadband. This is a limit-cycle.

Finite Wordlength Effects

- Consider the transfer function

$$G(z) = 1 \Big/ (1 + b_1 z^{-1} + b_2 z^{-2})$$

$$y_k = x_k - b_1 y_{k-1} - b_2 y_{k-2}$$

- if poles are complex then impulse response is given by h_k

$$h_k = \frac{\rho^k}{\sin\theta} . \sin\left[(k+1)\theta\right]$$

Finite Wordlength Effects

- Where $\rho = \sqrt{b_2}$ $\theta = \cos^{-1}\left({-b_1}\big/{2\sqrt{b_2}} \right)$
- If $b_2 = 1$ then the response is sinusiodal with frequency

$$\omega = \frac{1}{T}\cos^{-1}\left({-b_1}\big/{2} \right)$$

- Thus product quantisation causes instability implying an "effective " $b_2 = 1$.

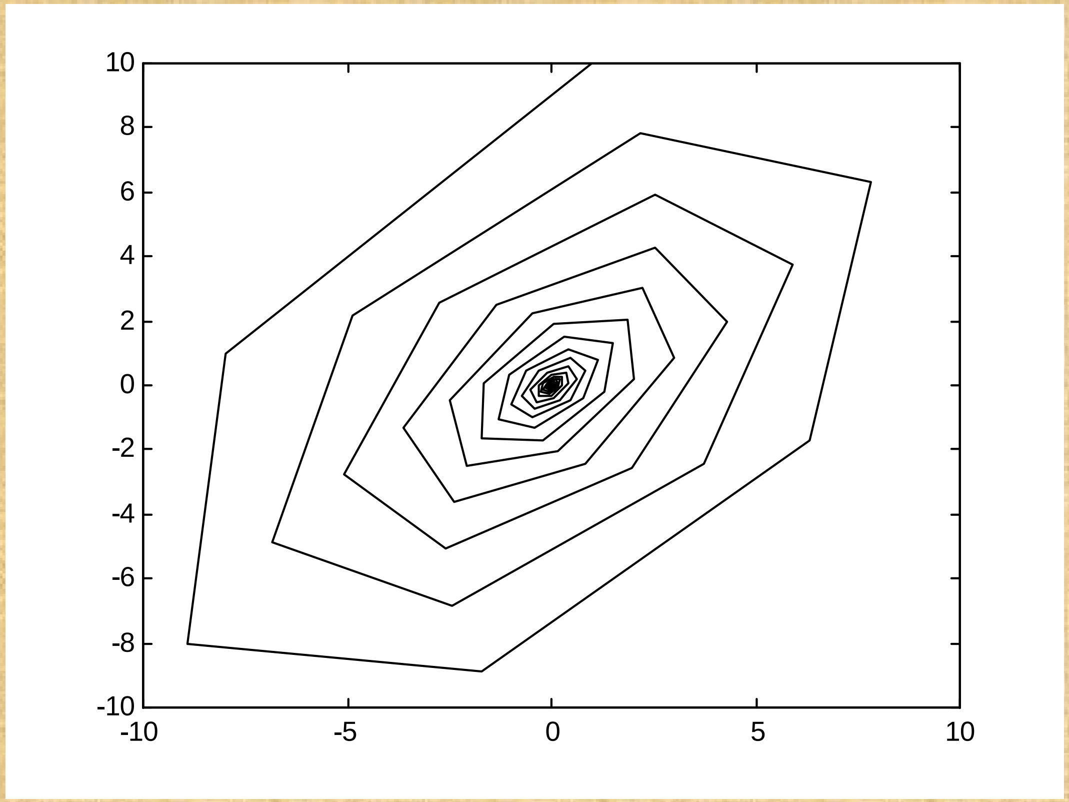
Finite Wordlength Effects

- Consider infinite precision computations for

$$y_k = x_k + y_{k-1} - 0.9\,y_{k-2}$$

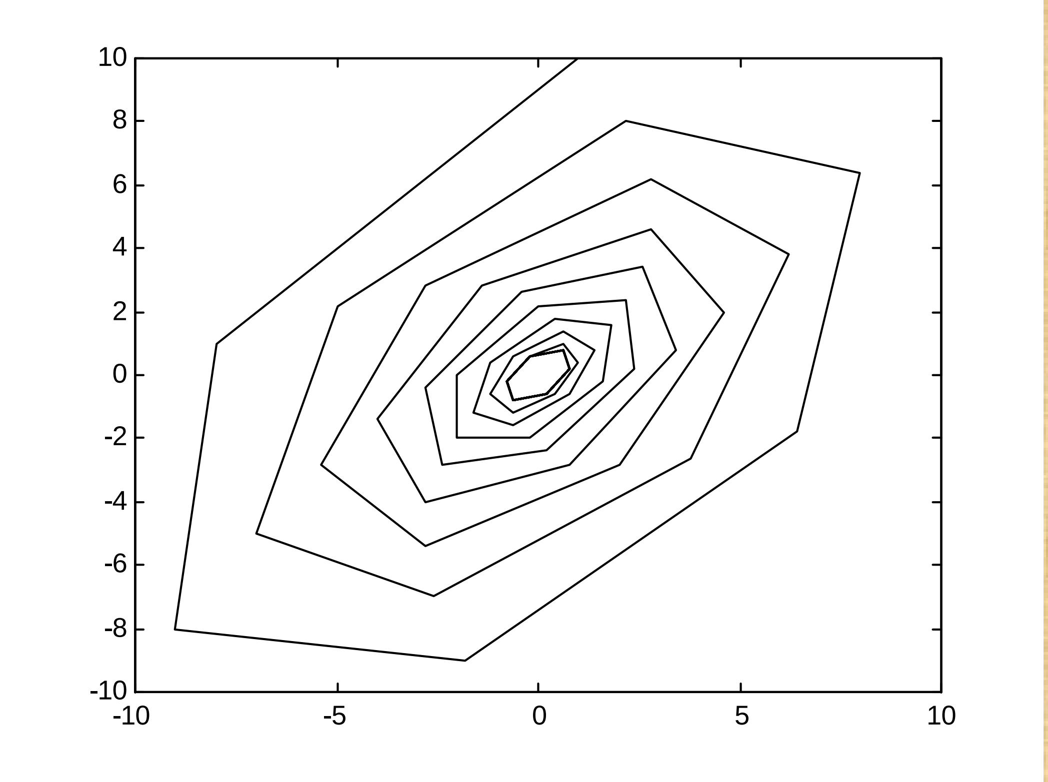
$$x_0 = 10$$

$$x_k = 0;\; k \neq 0$$



Finite Wordlength Effects

- Now the same operation with integer precision



Finite Wordlength Effects

- Notice that with infinite precision the response converges to the origin

- With finite precision the reponse does not converge to the origin but assumes cyclically a set of values –the Limit Cycle

Finite Wordlength Effects

- Assume $\{e_1(k)\}$,$\{e_2(k)\}$ ….. are not correlated, random processes etc.

$$\sigma_{0i}^{\ 2} = \sigma_e^{\ 2} \sum_{k=0}^{\infty} h_i^{\ 2}(k) \qquad \sigma_e^{\ 2} = Q^2 / 12$$

Hence total output noise power

$$\sigma_0^{\ 2} = \sigma_{01}^{\ 2} + \sigma_{02}^{\ 2} = 2.\frac{2^{-2b}}{12} \sum_{k=0}^{\infty} \rho^{2k} . \frac{\sin^2[(k+1)\theta]}{\sin^2 \theta}$$

- Where $Q = 2^{-b}$ and

$$h_1(k) = h_2(k) = \rho^k . \frac{\sin[(k+1)\theta]}{\sin\theta}; \ k \geq 0$$

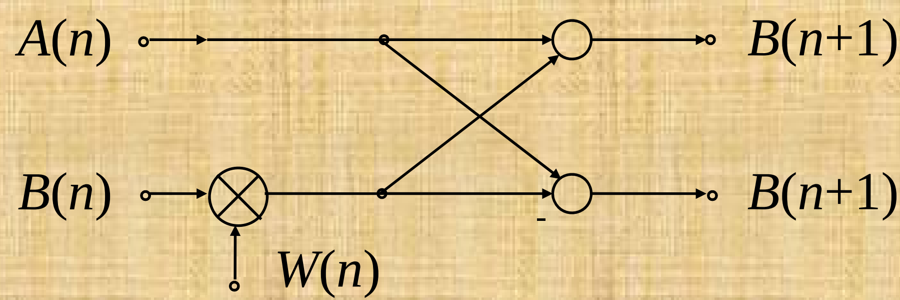
Finite Wordlength Effects

- ie

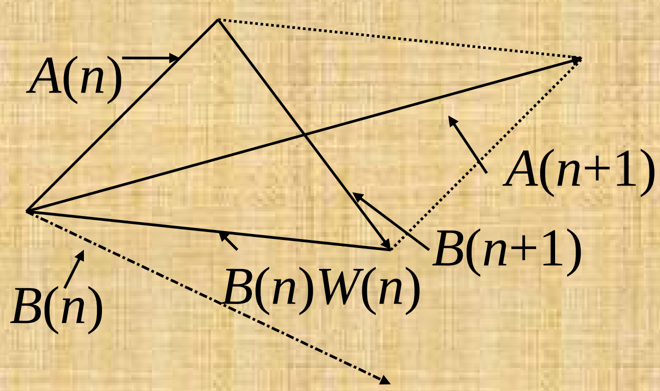
$$\sigma_0{}^2 = \frac{2^{-2b}}{6}\left[\frac{1+\rho^2}{1-\rho^2}\cdot\frac{1}{1+\rho^4-2\rho^2\cos 2\theta}\right]$$

Finite Wordlength Effects

- For FFT



$$A(n+1) = A(n) + W(n).B(n)$$
$$B(n+1) = A(n) - W(n).B(n)$$



Finite Wordlength Effects

- FFT

$$|A(n+1)|^2 + |B(n+1)|^2 = 2$$

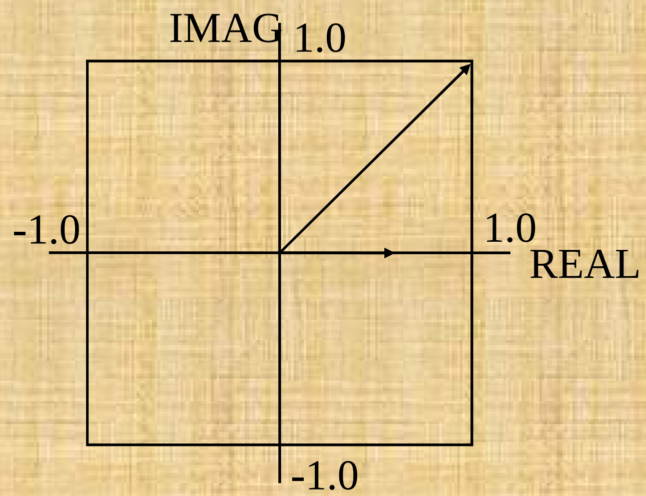
$$|A(n+1)|^2 = 2|A(n)|^2$$

$$|A(n)| = \sqrt{2}|A(n)|$$

- AVERAGE GROWTH: 1/2 BIT/PASS

Finite Wordlength Effects

- FFT



$$A_x(n+1) = A_x(n) + B_x(n)C(n) - B_y(n)S(n)$$

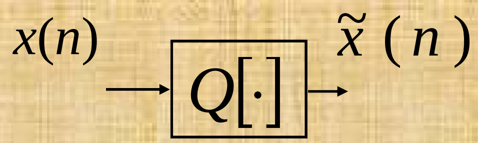
$$|A_x(n+1)| < |A_x(n)| + |B_x(n)||C(n)| - |B_y(n)||S(n)|$$

$$\frac{|A_x(n+1)|}{|A_x(n)|} < 1.0 + |C(n)| - |S(n)| = 2.414 \ldots.$$

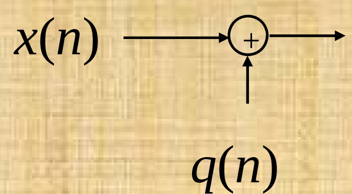
- PEAK GROWTH: 1.21.. BITS/PASS

Finite Wordlength Effects

- Linear modelling of product quantisation



- Modelled as

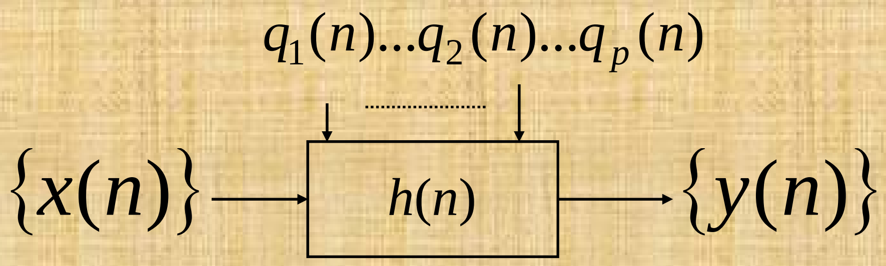


$$\tilde{x}(n) = x(n) + q(n)$$

Finite Wordlength Effects

- For rounding operations $q(n)$ is uniform distributed between $-\frac{Q}{2}, \frac{Q}{2}$ and where Q is the quantisation step (i.e. in a wordlength of bits with sign magnitude representation or mod 2, $Q = 2^{-b}$).
- A discrete-time system with quantisation at the output of each multiplier may be considered as a multi-input linear system

Finite Wordlength Effects



- Then

$$y(n) = \sum_{r=0}^{\infty} x(r).h(n-r) + \sum_{\lambda=1}^{p} \left[\sum_{r=0}^{\infty} q_{\lambda}(r).h_{\lambda}(n-r) \right]$$

- where $h_{\lambda}(n)$ is the impulse response of the system from λ the output of the multiplier to $y(n)$.

Finite Wordlength Effects

- For zero input i.e. $x(n) = 0, \forall n$ we can write

$$|y(n)| \leq \sum_{\lambda=1}^{p} |\hat{q}_\lambda| . \sum_{r=0}^{\infty} |h_\lambda(n-r)|$$

- where $|\hat{q}_\lambda|$ is the maximum of $|q_\lambda(r)|, \ \forall \lambda, r$ which is not more than $\frac{Q}{2}$

- ie $$|y(n)| \leq \frac{Q}{2} . \sum_{\lambda=1}^{p} \left[\sum_{n=0}^{\infty} |h_\lambda(n-r)| \right]$$

Finite Wordlength Effects

- However

$$\sum_{n=0}^{\infty} |h_\lambda(n)| \le \sum_{n=0}^{\infty} |h(n)|$$

- And hence

$$|y(n)| \le \frac{pQ}{2} . \sum_{n=0}^{\infty} |h(n)|$$

- ie we can estimate the maximum swing at the output from the system parameters and quantisation level